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On the stability of thermally radiative magnetofluiddynamic channel flow

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SUMMARY
The stability of three dimensional infinitesimal disturbances is examined for the laminar flow of a thermally radiating, viscous, electrically and heat-conducting fluid between parallel walls with transverse magnetic field. In addition to the classical criteria for the non-radiative case a further system of homogeneous differential equations with mixed boundary conditions arises yielding possible further eigenvalues and requirements for stable flows. For the overall configuration the analogue of Squire's theorem is shown to fail. The eigenvalue problem is of unusual type with the eigenvalue parameter appearing non linearly in both the differential equation and boundary conditions. A study of two-dimensional disturbances is carried out in detail for non-thermally conducting fluids and for a particular wave number the further eigenvalues are shown by exact analysis to be always stable. In channels with black walls at the same temperature extensive numerical calculations for a range of wave-numbers and a variety of other parameters all yield the same result.

1. Introduction

The stability of plane Poiseuille flow in hydrodynamics poses a classical problem and leads to the solution of the Orr-Sommerfeld equation, the analysis of which continues to be an active field of research, see Davey [1]. The effect of a magnetic field upon the stability of a corresponding magnetohydrodynamic flow was first studied in detail by Stuart [2] in the case when the magnetic field is parallel to the velocity and by Lock [3] for the case of a transverse field, commonly known as Hartmann flow. It was shown that in the latter configuration the magnetic field has a powerful stabilising influence, although the calculations were carried out only under conditions of small magnetic Reynolds number. More recent analysis by Sagalakov [4] and Potter and Kutchey [5] confirms Lock's results although there is shown to be a strong dependence additionally upon the magnetic Prandtl number.

In this paper a study is made of the stability of plane Hartmann flow for a fluid sufficiently hot for the effects of thermal radiation to be significant. Analysis of the steady state problem by the present author, Helliwell [6], in the case when thermal conductivity may be neglected leads to an exact closed form for the solution from which it is clear that thermal radiation has a marked effect upon the temperature profile. The same problem but with non-zero thermal conductivity was examined in a second paper, Helliwell [7]. However an analytic solution is now unavailable but from a numerical solution it is apparent that except in conditions of very strong conduction the previous distributions for the radiative flux and temperature remain qualitatively valid except in the immediate vicinity of the walls.

In the development of the stability problem effects of viscous and ohmic dissipation, thermal conductivity and radiation are retained together with three dimensional disturbances. In the calculations leading to the numerical results, however, in order that perturbations from an analytically expressed steady state may be studied the effects of thermal conductivity are neglected. Furthermore the disturbances are then restricted to be two dimensional. Whilst the neglect of molecular conduction at the same time as the retention of thermal radiation is apparently somewhat lacking in physical realism, previous investigations of stability problems involving thermal convection and radiative transfer have shown that the results are qualitatively the same whether or not molecular conduction is retained, see for instance papers by Christophorides and Davis [8] and Spiegel [9]. Thus for the present related problem the same assumption is perhaps not unreasonable. The calculations lead to the conclusion that thermally radiating magnetogasdynamic channel flows are never less stable than similar cooler flows in which the effects of radiative heat transfer may be neglected.

2. The basic equations and steady state

The governing equations for the flow of an electrically conducting viscous and heat conducting fluid are well established. They may be found, for instance, in the book by Shercliff [10]. The effects of thermal radiation at temperatures which are not extremely high are accounted for by the inclusion of an additional term in the energy equation. The features are discussed by Vincenti and Kruger [11] in their text upon the subject, and it is shown there that only the radiative flux is significant.

However the closure of the set of governing equations leads to a system of integro-differential equations which link the radiative flux to the radiative intensity and the equation of radiative transfer. Various forms of approximation have been employed in order to circumvent the rather complex nature of the exact system of equations. In the analysis of this paper the so-called differential approximation is used under conditions appropriate to a grey gas of arbitrary opacity, see for instance, Vincenti and Kruger (loc. cit.).

The steady state flow considered is that parallel to the $\bar{x}$ axis down a channel of great width in the $\bar{z}$ direction between walls distance $2h$ apart parallel to the $\bar{x}\bar{z}$ plane of a cartesian coordinate system. The bounds of the channel normal to the $\bar{z}$ axis are taken to be electrodes of perfect electrical conductivity whilst the walls normal to the $\bar{y}$ axis are supposed perfect insulators. An external magnetic field is applied uniformly across the channel in a direction normal to these insulators. The fluid in the channel is taken with a finite constant electrical conductivity and constant coefficients of viscosity, absorption and thermal conductivity, but is otherwise not defined further. Then under the assumption that the electrodes are infinitely far apart the problem becomes one-dimensional. As already remarked in the previous section, solutions have been presented in earlier papers by the present author for non-black walls of arbitrary emissivities and temperatures. The notation employed in these previous papers is retained for the analysis of this paper.

3. The perturbation equations

Without restriction three dimensional perturbations are considered. Steady state quantities are indicated with a bar superscript and are known functions of position, whilst perturbations are denoted by a tilde, thus $\boldsymbol{B} = \bar{\boldsymbol{B}} + \tilde{\boldsymbol{B}}$ where $\tilde{\boldsymbol{B}} = \boldsymbol{B}(\bar{x}, \bar{y}, \bar{z}, t)$. The following linearised forms of the governing equations may then be derived, where in the energy equation the notation of cartesian tensors is used in part. In general $\boldsymbol{B}$ denotes the magnetic flux, $\boldsymbol{V}$ the velocity, $\boldsymbol{q}$ the radiative flux, ρ the density, T the temperature, $\boldsymbol{j}$ the electric current density and $\boldsymbol{Z}$ the vorticity. Furthermore μ is the permeability, σ the electrical conductivity, η the coefficient of viscosity, k the thermal conductivity, α the volumetric absorption coefficient, $\tilde{\sigma}$ Stefan's constant, ε_1 and ε_2 the wall emissivities and c_v the specific heat at constant volume. The time is denoted by t.

It should first be noted however that for the steady state flow, as a consequence of the steady nature and one-dimensional configuration, as remarked above it is unnecessary to define the precise thermodynamic properties of the fluid. In particular the density is completely unrestricted and may be, for instance, a function of the temperature. However in order to ease and make tractable the analysis of many problems in thermal stability the assumption is made that the fluid is incompressible but subject to the Boussinesq approximation, see for example Christophorides and Davis [8]. Thus here the same broad assumption is introduced, which has the further advantage of making possible a direct comparison with previous work in the absence of radiation. It is not thought worthwhile in this initial study to introduce any additional complexity since the constancy of the various dissipative, diffusive and absorptive coefficients is itself a rather poor approximation. Further investigations are being undertaken for a more realistic model in which the dependence of these coefficients on the temperature is accounted for and in which no gross assumption is made concerning the compressibility. These will be reported in due course.

The conservation equation for the magnetic flux is

$$\operatorname{div} \tilde{\boldsymbol{B}} = 0. \tag{1}$$

The conservation equation for the mass is

$$\operatorname{div} \tilde{\boldsymbol{V}} = 0. \tag{2}$$

From the "magnetic vorticity" equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl}(\boldsymbol{V} \times \boldsymbol{B}) + \frac{1}{\mu\sigma} \nabla^2 \boldsymbol{B}$$

we derive

$$\frac{\partial \tilde{\boldsymbol{B}}}{\partial t} = (\tilde{\boldsymbol{B}} \cdot \operatorname{grad})\bar{\boldsymbol{V}} + (\bar{\boldsymbol{B}} \cdot \operatorname{grad})\tilde{\boldsymbol{V}} - (\bar{\boldsymbol{V}} \cdot \operatorname{grad})\tilde{\boldsymbol{B}} - (\tilde{\boldsymbol{V}} \cdot \operatorname{grad})\bar{\boldsymbol{B}} + \frac{1}{\mu\sigma} \nabla^2 \tilde{\boldsymbol{B}}. \tag{3}$$

From the momentum equation

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} + \rho(\boldsymbol{V} \cdot \operatorname{grad})\boldsymbol{V} - \boldsymbol{j} \times \boldsymbol{B} + \operatorname{grad} p - \eta \nabla^2 \boldsymbol{V} = 0$$

we obtain

$$\rho\left\{\frac{\partial\tilde{\boldsymbol{Z}}}{\partial t}+(\bar{\boldsymbol{V}}\cdot\text{grad})\tilde{\boldsymbol{Z}}+(\tilde{\boldsymbol{V}}\cdot\text{grad})\bar{\boldsymbol{Z}}-(\bar{\boldsymbol{Z}}\cdot\text{grad})\tilde{\boldsymbol{V}}-(\tilde{\boldsymbol{Z}}\cdot\text{grad})\bar{\boldsymbol{V}}\right\}$$

$$=\eta\nabla^2\tilde{\boldsymbol{Z}}+(\bar{\boldsymbol{B}}\cdot\text{grad})\tilde{\boldsymbol{j}}+(\tilde{\boldsymbol{B}}\cdot\text{grad})\bar{\boldsymbol{j}}-(\bar{\boldsymbol{j}}\cdot\text{grad})\tilde{\boldsymbol{B}}-(\tilde{\boldsymbol{j}}\cdot\text{grad})\bar{\boldsymbol{B}}, \tag{4}$$

where

$$\bar{\boldsymbol{Z}}=\text{curl}\,\bar{\boldsymbol{V}},\qquad \tilde{\boldsymbol{Z}}=\text{curl}\,\tilde{\boldsymbol{V}}, \tag{5}$$

$$\mu\bar{\boldsymbol{j}}=\text{curl}\,\bar{\boldsymbol{B}},\qquad \mu\tilde{\boldsymbol{j}}=\text{curl}\,\tilde{\boldsymbol{B}}. \tag{6}$$

The energy equation

$$\rho c_v\left(\frac{\partial T}{\partial t}+\boldsymbol{V}\cdot\text{grad}\,T\right)+\text{div}\,\boldsymbol{q}-k\nabla^2T-\frac{j^2}{\sigma}-\eta V_{l,m}(V_{l,m}+V_{m,l})=0$$

becomes

$$\rho c_v\left(\frac{\partial\tilde{T}}{\partial t}+\bar{\boldsymbol{V}}\cdot\text{grad}\,\tilde{T}+\tilde{\boldsymbol{V}}\cdot\text{grad}\,\bar{T}\right)+\text{div}\,\tilde{\boldsymbol{q}}-k\nabla^2\tilde{T}-\frac{2}{\sigma}\bar{\boldsymbol{j}}\cdot\tilde{\boldsymbol{j}}$$

$$-\eta\tilde{V}_{l,m}(\bar{V}_{m,l}+2\bar{V}_{l,m})-\eta\tilde{V}_{m,l}\bar{V}_{l,m}=0 \tag{7}$$

The equation for the radiative flux

$$\text{grad div}\,\boldsymbol{q}-3\alpha^2\boldsymbol{q}-4\tilde{\sigma}\alpha\,\text{grad}\,(T^4)=0$$

takes the form

$$\text{grad div}\,\tilde{\boldsymbol{q}}-3\alpha^2\tilde{\boldsymbol{q}}-16\tilde{\sigma}\alpha(\bar{T}^3\,\text{grad}\,\tilde{T}+3\bar{T}^2\tilde{T}\,\text{grad}\,\bar{T})=0. \tag{8}$$

The system of equations (1) to (6) is the basic set previously studied by Lock [3] in the absence of radiative effects. The reader should note that certain of these equations are implied by the others and that no difficulties result from the fact that there appear to be more equations than variables. Stuart [2] commented explicitly upon the matter in relation to the analogous set of equations in his paper.

It remains to specify the boundary conditions upon the perturbed quantities. These relate to electromagnetic and no-slip conditions at the walls together with radiative conditions, which for walls of general emissivity have been given by Cess [12] under the differential approximation. With the $\bar{x}$ axis taken to be the centre line of the channel, the perturbed forms are

$$\tilde{\boldsymbol{V}}=0,\quad \tilde{j}_y=0,\quad \tilde{B}_y=0,\quad \tilde{T}=0;\quad \bar{y}=\pm h, \tag{9}$$

$$16\tilde{\sigma}\bar{T}^3\tilde{T}-\left(\frac{4}{\varepsilon_2}-2\right)\tilde{q}_y-\frac{1}{\alpha}\text{div}\,\tilde{\boldsymbol{q}}=0;\quad \bar{y}=h,$$

$$16\tilde{\sigma}\bar{T}^3\tilde{T}+\left(\frac{4}{\varepsilon_1}-2\right)\tilde{q}_y-\frac{1}{\alpha}\text{div}\,\tilde{\boldsymbol{q}}=0;\quad \bar{y}=-h, \tag{10}$$

in the case $k \neq 0$. If thermal conductivity is negligible so that $k = 0$ then the condition $\tilde{T} = 0$ must be relaxed at both boundaries.

When the perturbation equations are written in non-dimensional form a number of dimensionless parameters arise, conventionally named as follows:

Hartmann No.	$M = B_0 h\left(\dfrac{\sigma}{\eta}\right)^{\frac{1}{2}}$,	Bouger No.	$\omega = \alpha h$,
Reynolds No.	$R = \dfrac{\rho U_0 h}{\eta}$,	Boltzmann No.	$\beta = \dfrac{\rho U_0^3}{\tilde{\sigma} T_1^4}$,
Magnetic Reynolds No.	$R_m = \mu\sigma U_0 h$,	Prandtl No.	$P_r = \dfrac{\eta c_p}{k}$,
Adiabatic Index	$\gamma = \dfrac{c_p}{c_v}$,	Eckert No.	$\epsilon = \dfrac{U_0^2}{c_p T_1}$.

We also define

Temperature Ratio	$\Theta = \dfrac{T_2}{T_1}$,	Current No.	$J = \dfrac{j_0}{\sigma U_0 B_0}$

where B_0 is the applied magnetic flux, U_0 and j_0 are the mean steady state fluid speed and current density respectively, T_1 and T_2 are the wall temperatures at $y = \mp h$ respectively, and c_p the specific heat at constant pressure. The associated non-dimensional variables are

Coordinates:	$x, y, z = \dfrac{\bar{x}}{h}, \dfrac{\bar{y}}{h}, \dfrac{\bar{z}}{h}$	Time:	$\tau = \dfrac{U_0 t}{h}$
Velocity:	$\boldsymbol{v} = \dfrac{\boldsymbol{V}}{U_0}$	Magnetic Flux:	$\boldsymbol{b} = \dfrac{\boldsymbol{B}}{B_0}$
Radiative Flux:	$\boldsymbol{Q} = \dfrac{\boldsymbol{q}}{\sigma T_1^4}$	Temperature:	$\theta = \dfrac{T}{T_1}$

Three dimensional perturbations are considered of the form

$$\tilde{\boldsymbol{v}} = \{v_x(y), v_y(y), v_z(y)\} \exp i[K_1(x - c\tau) + K_3 z], \tag{11}$$

$$\tilde{\boldsymbol{b}} = \{b_x(y), b_y(y), b_z(y)\} \exp i[K_1(x - c\tau) + K_3 z], \tag{12}$$

$$\tilde{\boldsymbol{Q}} = \{Q_x(y), Q_y(y), Q_z(y)\} \exp i[K_1(x - c\tau) + K_3 z], \tag{13}$$

$$\tilde{\theta} = \theta(y) \exp i[K_1(x - c\tau) + K_3 z]. \tag{14}$$

It will be found useful to set

$$K^2 = K_1^2 + K_3^2. \tag{15}$$

Equations (1)–(6) together with the appropriate boundary conditions separate from the remainder and generate relationships between the velocity and magnetic flux alone. After considerable algebra these yield six independent differential equations which may be written

$$\left(\frac{d}{dy} + iK_1 b_1\right) v_x - b_1' v_y + \left\{\frac{1}{R_m}\frac{d^2}{dy^2} - \frac{K^2}{R_m} + iK_1(c - v_1)\right\} b_x + v_1' b_y = 0, \tag{16}$$

$$\left(\frac{d}{dy} + iK_1 b_1\right) v_y + \left\{\frac{1}{R_m}\frac{d^2}{dy^2} - \frac{K^2}{R_m} + iK_1(c - v_1)\right\} b_y = 0, \tag{17}$$

$$iK_1 b_x + \frac{db_y}{dy} + iK_3 b_z = 0, \tag{18}$$

$$iK_1 v_x + \frac{dv_y}{dy} + iK_3 v_z = 0, \tag{19}$$

$$\begin{aligned} &\left\{\frac{iK_3}{R}\frac{d^2}{dy^2} - \frac{iK^2K_3}{R} - K_1K_3(c - v_1)\right\} v_y \\ &\quad + \left\{-\frac{1}{R}\frac{d^3}{dy^3} + \left[\frac{K^2}{R} - iK_1(c - v_1)\right]\frac{d}{dy} + iK_1 v_1'\right\} v_z \\ &\quad + \frac{M^2}{RR_m} iK_3\left(\frac{d}{dy} + iK_1 b_1\right) b_y + \frac{M^2}{RR_m}\left\{-\frac{d^2}{dy^2} - iK_1\frac{d}{dy} - iK_1 b_1'\right\} b_z = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} &\left\{-\frac{iK_3}{R}\frac{d^2}{dy^2} + \frac{iK^2K_3}{R} + K_1K_3(c - v_1)\right\} v_x \\ &\quad + iK_3 v_1' v_y + \left\{\frac{iK_1}{R}\frac{d^2}{dy^2} - \frac{iK^2K_1}{R} - K_1^2(c - v_1)\right\} v_z \\ &\quad - \frac{M^2}{RR_m} iK_3\left(\frac{d}{dy} + iK_1 b_1\right) b_x - \frac{M^2}{RR_m} iK_3 b_1' b_y \\ &\quad + \frac{M^2}{RR_m} iK_1\left(\frac{d}{dy} + iK_1 b_1\right) b_z = 0. \end{aligned} \tag{21}$$

Here the steady state variables, v_1, b_1 are given by, see Helliwell [6],

$$v_1 = \frac{M(\cosh M - \cosh My)}{M\cosh M - \sinh M}, \tag{22}$$

$$b_1 = R_m\left\{\frac{\sinh My}{M\cosh M - \sinh M} - y\left(\frac{\sinh M}{M\cosh M - \sinh M} + J\right)\right\}, \tag{23}$$

and a prime denotes the derivative with respect to y. The system may be reduced to a pair of differential equations in b_y and v_y alone. These are equation (17) together with

$$\begin{aligned} &\left\{\frac{i}{R}\frac{d^4}{dy^4} - \left[\frac{2iK^2}{R} + K_1(c - v_1)\right]\frac{d^2}{dy^2} + K^2\left[\frac{iK^2}{R} + K_1(c - v_1)\right] - K_1 v_1''\right\} v_y \\ &\quad + \frac{M^2}{RR_m}\left\{i\frac{d^3}{dy^3} - K_1 b_1\frac{d^2}{dy^2} - iK^2\frac{d}{dy} + K_1K^2 b_1 + K_1 b_1''\right\} b_y = 0. \end{aligned} \tag{24}$$

Equations analogous to these have been studied in the analysis of the non-radiative problem, see Lock [3] equations (21) and (22), and the reader is referred to his paper and those by Sagalakov [4] and Potter and Kutchey [5] for the details.

The radiative effects are restricted to equations (7) and (8) with the relevant boundary conditions. One obtains four equations relating $v_x, v_y, b_x, b_y, \theta, Q_x, Q_y, Q_z$. From these Q_x and Q_z may be eliminated to leave the two remaining equations

$$\left\{(K^2 + 3\omega^2)\left[iK_1(v_1 - c) + \frac{\gamma K^2}{RP_r} - \frac{\gamma}{RP_r}\frac{d^2}{dy^2}\right] + \frac{16\gamma\epsilon K^2\omega\theta_1^3}{\beta}\right\}\theta$$
$$+ \frac{3\gamma\epsilon\omega^2}{\beta}\frac{dQ_y}{dy} + (K^2 + 3\omega^2)\left\{-\frac{2\gamma\epsilon v_1'}{R}\frac{dv_x}{dy} + \left(\theta_1' - \frac{2\gamma\epsilon iK_1 v_1'}{R}\right)v_y\right.$$
$$\left. - \frac{2\gamma M^2\epsilon b_1'}{RR_m^2}\frac{db_x}{dy} + \frac{2\gamma M^2\epsilon iK_1 b_1'}{RR_m^2}b_y\right\} = 0, \tag{25}$$

$$3\omega\left\{\frac{d^2}{dy^2} - (K^2 + 3\omega^2)\right\}Q_y + 16(K^2 - 1)\theta_1^2\left(\theta_1\frac{d}{dy} + 3\theta_1'\right)\theta = 0, \tag{26}$$

where θ_1 is the steady state dimensionless temperature and v_x, v_y, b_x, b_y may be supposed known solutions of the separated system (16)–(24). Therefore a system of six equations governing the stability problem for radiative flow may be taken as equations (16, 17, 24, 25, 26) together with the following equation (27) obtained from equations (18), (19) and (21)

$$\left\{\frac{K^2}{R}\frac{d^2}{dy^2} - \frac{K^4}{R} + iK_1K^2(c - v_1)\right\}v_x$$
$$+ \left\{-\frac{iK_1}{R}\frac{d^3}{dy^3} + \left[\frac{iK^2K_1}{R} + K_1^2(c - v_1)\right]\frac{d}{dy} - K_3^2 v_1'\right\}v_y$$
$$+ \frac{M^2}{RR_m}K^2\left(\frac{d}{dy} + iK_1b_1\right)b_x$$
$$+ \frac{M^2}{RR_m}\left\{-iK_1\frac{d^2}{dy^2} + K_1^2 b_1\frac{d}{dy} + K_3^2 b_1'\right\}b_y = 0. \tag{27}$$

Thus with radiative effects present additional equations arise governing the flow from which further conditions for stability may arise. Thermal radiation therefore cannot provide a stabilizing influence, although its effect may be neutral should the disturbances associated with equations (25) and (26) be always stable.

4. Two- and three dimensional disturbances: Squire's theorem

The disturbances are two-dimensional if $K_3 = 0$ and then $K = K_1$. Otherwise, for three-dimensional disturbances, $K > K_1$. In the absence of both thermal radiation and magnetic field Squire (13) showed that two-dimensional disturbances become unstable at lower values of Reynolds number than similar three-dimensional disturbances. Subsequently Lock [3] proved that Squire's theorem remains true in the presence of a transverse magnetic field provided that the comparison is made with M held fixed. However in the case of an aligned magnetic field Hunt [14] demonstrated that the theorem is no longer true. The validity of the theorem is now considered for thermally radiating flows with transverse field.

In the governing equations (16, 17, 24, 25, 26, 27) retain M invariant and set

$$K_1 R = K\bar{R}, \quad K_1 R_m = K\bar{R}_m, \quad K_1 b_1 = K\bar{b}_1, \quad K_1 b_x = K\bar{b}_x, \quad K_1 b_y = K\bar{b}_y$$

where, meantime, the bar denotes a quantity in the two-dimensional problem. It turns out that equations (17) and (24) alone may be written entirely in terms of K and the barred quantities. On the other hand the two equations (25) and (26) containing Q_y and θ cannot both be so written. Thus it follows that no firm conclusion may be reached concerning the relative stability of two- and three-dimensional disturbances and Squire's theorem is not proven for radiating flows. This conclusion is unaffected by the values of R_m and P_r and so remains true in both purely fluidynamic and non-thermally conducting flows with thermal radiation. However should it prove to be the case that equations (25) and (26) relate to disturbances always stable then their effect may be discounted and Squire's theorem in consequence would remain true. This may well be the case since the analysis of the following sections relating to two-dimensional disturbances in flows without thermal conductivity does indicate that then the perturbations Q_y and θ are always stable.

5. Two-dimensional disturbances without thermal conductivity

As a first study only two-dimensional disturbances without z-variation are considered, so that $K_1 = K$. Further since in the determination of the steady state explicit forms of solution are available only in the absence of thermal conductivity, the stability analysis is restricted to this case with $P_r \to \infty$.

In the situation in which perturbations v_y and b_y arise the stability criteria arise from the homogeneous equations (17) and (24) alone which, as remarked earlier, have been analysed by Lock and others. Should however v_y and b_y be identically zero these two equations vanish but equations (16) and (27) for v_x and b_x now become homogeneous. These equations are however satisfied identically, for, from equations (18) and (19) with $v_z \equiv 0$, $b_z \equiv 0$ it is then necessary that v_x and b_x be also identically zero. Hence no additional stability criteria are obtained.

When thermal radiation is present, should only disturbances in the radiative flux and temperature occur the final pair of equations (25) and (26) become homogeneous and supplementary stability criteria may arise from their solution. The variable Q_y may be eliminated from them to yield a single equation for θ. It is convenient to introduce a new variable ϕ given by

$$\phi = \left\{ iK(v_1 - c) + \frac{16\gamma\epsilon\omega\theta_1^3}{\beta(K^2 + 3\omega^2)} \right\} \theta \tag{28}$$

and then the governing differential equation becomes

$$\frac{d^2\phi}{dy^2} - (K^2 + 3\omega^2) \left\{ \frac{i\beta K(v_1 - c)(K^2 + 3\omega^2) + 16\gamma\epsilon\omega K^2\theta_1^3}{i\beta K(v_1 - c)(K^2 + 3\omega^2) + 16\gamma\epsilon\omega\theta_1^3} \right\} \phi = 0, \tag{29}$$

where v_1 is given by equation (22) and, see Helliwell [6],

$$\theta_1^4 = \frac{\frac{1}{2}}{(3\omega - 2)\varepsilon_1\varepsilon_2 + 2(\varepsilon_1 + \varepsilon_2)}$$

$$\times \{[(3\omega - 2)\varepsilon_2 + 4]\varepsilon_1 + [(3\omega - 2)\varepsilon_1 + 4]\varepsilon_2\Theta^4 + 3\omega y\varepsilon_1\varepsilon_2(\Theta^4 - 1)\}$$

$$- \frac{MP}{2N\omega(M\cosh M - \sinh M)}\{(M^2 - 3\omega^2)\cosh My + 3\omega^2\cosh M\}$$

$$- \frac{M^2P^2}{4N\omega}\{\tfrac{3}{2}\omega^2(y^2 - 1) - 1\}$$

$$+ \frac{M^2}{16N\omega(M\cosh M - \sinh M)^2}\{(4M^2 - 3\omega^2)\cosh 2My + 3\omega^2\cosh 2M\}$$

$$+ \frac{M^2}{2N}\left\{\frac{3\omega y(\varepsilon_1 - \varepsilon_2) + 3\omega(\varepsilon_1 + \varepsilon_2 - \varepsilon_1\varepsilon_2) + 2(2\varepsilon_1 - \varepsilon_2)(2\varepsilon_2 - \varepsilon_1)}{(3\omega - 2)\varepsilon_1\varepsilon_2 + 2(\varepsilon_1 + \varepsilon_2)}\right\}$$

$$\times \left\{P^2 - \frac{2P\sinh M}{M\cosh M - \sinh M} + \frac{M\sinh 2M}{2(M\cosh M - \sinh M)^2}\right\} \tag{30}$$

with

$$N = \frac{R}{\beta}, \quad P = J + \frac{\sinh M}{M\cosh M - \sinh M}.$$

The boundary conditions are derived from equations (9) and (10) with the relaxation of $\theta = 0$ at $y = \pm 1$. Equation (8) may be used to eliminate Q_x and Q_z. Equations (25) and (26) are used to eliminate Q_y so that the conditions relate to θ alone. Thus, using equation (28) and noting that $v_1 = 0$ at $y = \pm 1$ the final form becomes

$$\left(\frac{2}{\varepsilon_\pm} - 1\right)\frac{d\phi}{dy} \pm \frac{3\omega}{2}(K^2 + 3\omega^2)\left\{\frac{16\gamma\epsilon\omega\theta_{w\pm}^3 - iKc\beta}{16\gamma\epsilon\omega\theta_{w\pm}^3 - iKc\beta(K^2 + 3\omega^2)}\right\}\phi = 0, \quad y = \pm 1, \tag{31}$$

where $\varepsilon_\pm$ is the value of the emissivity and $\theta_{w\pm}$ is the value of θ_1 at the appropriate boundary.

The criterion for stable disturbances is that $\text{Imag}(c) < 0$.

Whilst for general wave-number K a solution of equations (29) with (31) can only be obtained by numerical analysis, an exact solution is available when $K = 1$ and so in this instance the eigenvalue c may be obtained explicitly as a function of the parameters. For convenience set $C = i\beta c$ and it then turns out that C satisfies the equation

$$\{(fgk^6 + m^2)\sinh 2k + (f + g)mk^3\cosh 2k\}C^2$$
$$- \{(X_+ + X_-)(fgk^4 + m^2)\sinh 2k + [X_+(f + gk^2) + X_-(g + fk^2)]km\cosh 2k\}C$$
$$+ \{(fgk^2 + m^2)\sinh 2k + (f + g)mk\cosh 2k\}X_+X_- = 0,$$

where $X_\pm = 16\gamma\epsilon\omega\theta_{w\pm}^3$, $k = (1 + 3\omega^2)^{\frac{1}{2}}$, $m = \frac{3}{2}\omega(1 + 3\omega^2)$, $f = (2/\varepsilon_2) - 1$, $g = (2/\varepsilon_1) - 1$.

This is a quadratic equation of the form $a_0C^2 - a_1C + a_2 = 0$ with $a_0, a_1, a_2 > 0$ and thus Real$(C) > 0$ necessarily. It follows therefore that Imag$(c) < 0$ and the disturbances are always stable.

6. Numerical solutions

Symmetric configurations with black walls at the same temperature are considered. Then $\varepsilon_1 = \varepsilon_2 = 1$ and $\Theta = 1$ so that both v_1 and θ_1 are even functions of y. When $K = 1$ one finds that eigenfunctions ϕ relate to either symmetric or antisymmetric disturbances for which the corresponding eigenvalues are purely imaginary and such that, respectively

$$\text{Imag}(c) = -\frac{16\gamma\epsilon\omega\theta_w^3(2\sinh\Omega + 3\omega\Omega\cosh\Omega)}{\beta\Omega(2\Omega\sinh\Omega + 3\omega\cosh\Omega)}$$

or,

$$\text{Imag}(c) = -\frac{16\gamma\epsilon\omega\theta_w^3(2\cosh\Omega + 3\omega\Omega\sinh\Omega)}{\beta\Omega(2\Omega\cosh\Omega + 3\omega\sinh\Omega)}$$

where $\Omega = (1 + 3\omega^2)^{\frac{1}{2}}$ and θ_w is the common value of $\theta_{w\pm}$. For general $K \neq 1$ equation (29) and conditions (31) may then be advantageously remodelled for purposes of numerical solution. Odd and even eigenfunctions may be separated so that each satisfies equation (29) but the domain becomes the interval $0 \leqq y \leqq 1$ with boundary conditions

at $y = 1$,
$$\frac{d\phi}{dy} + \tfrac{3}{2}\omega(K^2 + 3\omega^2)\left\{\frac{16\gamma\epsilon\omega\theta_w^3 - iKc\beta}{16\gamma\epsilon\omega\theta_w^3 - iKc\beta(K^2 + 3\omega^2)}\right\}\phi = 0, \tag{32}$$

and, at $y = 0$,
$$\phi = 0 \quad \text{for odd eigenfunctions,} \tag{33}$$

$$\frac{d\phi}{dy} = 0 \text{ for even eigenfunctions.} \tag{34}$$

Let the eigenvalue be written explicitly $c = a + ib$ with corresponding eigenfunction $\phi = U + iV$. Then the governing equation (29) may be written as the system of four first order linear equations

$$\begin{aligned} \frac{dU}{dy} &= W, \quad \frac{dW}{dy} = fU - gV, \\ \frac{dV}{dy} &= Z, \quad \frac{dZ}{dy} = gU + fV, \end{aligned} \tag{35}$$

where

$$\begin{aligned} f &= C\{[Kb\beta(K^2 + 3\omega^2) + 16\gamma\epsilon\omega\theta_1^3][Kb\beta(K^2 + 3\omega^2) + 16\gamma\epsilon\omega K^2\theta_1^3] \\ &\quad + K^2\beta^2(K^2 + 3\omega^2)^2(v_1 - a)^2\} \\ g &= 16C\gamma\epsilon\omega K\beta(1 - K^2)(K^2 + 3\omega^2)\theta_1^3(v_1 - a), \\ C &= (K^2 + 3\omega^2)/\{[Kb\beta(K^2 + 3\omega^2) + 16\gamma\epsilon\omega\theta_1^3]^2 + K^2\beta^2(K^2 + 3\omega^2)^2(v_1 - a)^2\}. \end{aligned}$$

The boundary conditions become

$$\text{at } y = 1, \qquad EW + FZ + GU + HV = 0,$$

$$EZ - FW + GV + HU = 0, \tag{36}$$

$$\text{and at } y = 0, \quad U = 0,\ V = 0 \text{ for odd eigenfunctions,}$$

$$W = 0,\ Z = 0 \text{ for even eigenfunctions,}$$

where

$$E = Kb\beta(K^2 + 3\omega^2) + 16\gamma\epsilon\omega\theta_w^3, \qquad F = Ka\beta(K^2 + 3\omega^2),$$

$$G = \tfrac{3}{2}\omega(K^2 + 3\omega^2)(Kb\beta + 16\gamma\epsilon\omega\theta_w^3), \quad H = \tfrac{3}{2}\omega(K^2 + 3\omega^2)Ka\beta.$$

In the case of odd eigenfunctions one solution of equations (35) is determined, viz. $U = U^{(1)}$, $V = V^{(1)}$, $W = W^{(1)}$, $Z = Z^{(1)}$, fitting the boundary conditions $U = 0$, $V = 0$, $W = 1$, $Z = 0$ at $y = 0$, and a second solution is also obtained, viz. $U = U^{(2)}$, $V = V^{(2)}$, $W = W^{(2)}$, $Z = Z^{(2)}$ fitting the alternative conditions $U = 0$, $V = 0$, $W = 0$, $Z = 1$ at $y = 0$. It is a simple matter to show analytically that $U^{(2)} = -V^{(1)}$, $V^{(2)} = U^{(1)}$, $W^{(2)} = -Z^{(1)}$, $Z^{(2)} = W^{(1)}$. A linear combination of these solutions is then derived to fit the conditions (36). From this it transpires that

$$\Phi = EW_1^{(1)} + FZ_1^{(1)} + GU_1^{(1)} + HV_1^{(1)} = 0,$$

$$\Psi = EZ_1^{(1)} - FW_1^{(1)} + GV_1^{(1)} - HU_1^{(1)} = 0,$$

where suffix (1) denotes the value of the function computed at $y = 1$. This pair of equations will be identically satisfied only if the initial choice of a and b has been made correctly in the determination of the solution $U = U^{(1)}$, etc. Thus they are used to form the basis of an interpolation and iteration to find a and b as follows.

An estimate is made of (a, b) and $\Phi(a, b)$, $\Psi(a, b)$ computed. A second pair of values $(a + \delta a, b)$ are taken and $\Phi(a + \delta a, b)$, $\Psi(a + \delta a, b)$ computed, and a third pair $(a, b + \delta b)$ taken and $\Phi(a, b + \delta b)$, $\Psi(a, b + \delta b)$ evaluated. From these estimates for $\partial\Phi/\partial a$, $\partial\Phi/\partial b$, $\partial\Psi/\partial a$, $\partial\Psi/\partial b$ are obtained. Hence by assuming that the correct pair are $(a + \Delta a, b + \Delta b)$ approximate values for Δa and Δb are derived from the solution of

$$\Phi(a, b) + \frac{\partial\Phi}{\partial a}\Delta a + \frac{\partial\Phi}{\partial b}\Delta b = 0,$$

$$\Psi(a, b) + \frac{\partial\Psi}{\partial a}\Delta a + \frac{\partial\Psi}{\partial b}\Delta b = 0.$$

For even eigenvalues a similar procedure is followed with the solution $U^{(1)}$, etc. satisfying the boundary conditions $U = 1$, $V = 0$, $W = 0$, $Z = 0$ at $y = 0$.

In both cases the calculations are started from the known exact solution for (a, b) at $K = 1$ and the value of K changed progressively from this base. The following tables give the results for various ranges of values of the parameters. In every instance $\mathrm{Imag}(c) < 0$ so that disturbances in the radiative flux and temperature alone are always stable.

However it should be noted that the eigenvalue problem is not of simple type. The eigenvalue, c, appears non-linearly in both the differential equation and the boundary conditions. Such problems have appeared in the literature, but with the eigenvalue present only in the differential equation, see for instance Kelly [15]. In such cases a discrete spectrum of eigenvalues usually exists. Thus, for the present problem, although a unique eigenvalue occurs when $K = 1$, one cannot be certain that the single eigenvalue for each ($K \neq 1$) as determined by the numerical procedure is the only one possible. Further analysis of this aspect of the problem is necessary, but this is not pursued here.

TABLE 1

Imag (c). Poiseuille flow

	Odd eigenvalues						Even eigenvalues		
ω	1.0			0.1			0.1		
R	10^3	10^5	10^7	10^3	10^5	10^7	10^3	10^5	10^7
K									
0.2	−0.264	−0.086	−0.083	−1.88	−0.198	−0.168	−0.546	−0.075	−0.063
0.4	−0.132	−0.042	−0.041	−0.353	−0.037	−0.032	−0.238	−0.026	−0.022
0.6	−0.086	−0.028	−0.027	−0.118	−0.012	−0.011	−0.103	−0.011	−0.009
0.8	−0.063	−0.020	−0.020	−0.054	−0.006	−0.005	−0.052	−0.005	−0.005
1.0	−0.049	−0.016	−0.015	−0.029	−0.003	−0.003	−0.029	−0.003	−0.003
1.2	−0.039	−0.013	−0.012	−0.018	−0.002	−0.002	−0.018	−0.002	−0.002
1.4	−0.032	−0.010	−0.010	−0.012	−0.001	−0.001	−0.012	−0.001	−0.001
1.6	−0.027	−0.009	−0.009	−0.008	−0.001	−0.001	−0.008	−0.001	−0.001
1.8	−0.023	−0.007	−0.007	−0.006	−0.001	−0.001	−0.006	−0.001	−0.001

$\gamma = {}^5/_3, \ \varepsilon = 1, \ \beta = 1000, \ M = 0.$

TABLE 2

Imag (c). Odd eigenvalues. Hartmann flow

J	−0.5			0			0.5			1.0		
R	10^3	10^5	10^7	10^3	10^5	10^7	10^3	10^5	10^7	10^3	10^5	10^7
K												
0.2	−11.2	−0.425	−0.171	−7.73	−0.348	−0.170	−6.13	−0.304	−0.170	−5.91	−0.297	−0.170
0.4	−1.97	−0.080	−0.032	−1.45	−0.065	−0.032	−1.15	−0.057	−0.032	−1.11	−0.056	−0.032
0.6	−0.657	−0.027	−0.011	−0.486	−0.022	−0.011	−0.384	−0.019	−0.011	−0.369	−0.019	−0.011
0.8	−0.298	−0.012	−0.005	−0.220	−0.010	−0.005	−0.175	−0.009	−0.005	−0.168	−0.009	−0.005
1.0	−0.161	−0.007	−0.003	−0.119	−0.005	−0.003	−0.095	−0.005	−0.003	−0.091	−0.005	−0.003
1.2	−0.098	−0.004	−0.002	−0.073	−0.003	−0.002	−0.058	−0.003	−0.001	−0.055	−0.003	−0.002
1.4	−0.065	−0.003	−0.001	−0.048	−0.002	−0.001	−0.038	−0.002	−0.001	−0.036	−0.002	−0.001
1.6	−0.045	−0.002	−0.001	−0.033	−0.002	−0.001	−0.026	−0.001	−0.001	−0.026	−0.001	−0.001
1.8	−0.033	−0.001	−0.001	−0.025	−0.001	−0.001	−0.019	−0.001	−0.001	−0.019	−0.001	−0.001

$\gamma = {}^5/_3, \ \varepsilon = 1, \ \beta = 1000, \ \omega = 0.1, \ M = 5.$

TABLE 3

Image (c). Odd eigenvalues. Hartmann flow

β	100			10000		
R	10^3	10^5	10^7	10^3	10^5	10^7
K						
0.2	−11.4	−1.83	−1.68	−3.43	−0.115	−0.018
0.4	−2.17	−0.344	−0.317	−0.643	−0.021	−0.003
0.6	−0.724	−0.155	−0.106	−0.215	−0.007	−0.001
0.8	−0.328	−0.052	−0.048	−0.098	−0.003	−0.001
1.0	−0.178	−0.028	−0.026	−0.053	−0.002	−0.000
1.2	−0.108	−0.017	−0.016	−0.032	−0.001	−0.000
1.4	−0.071	−0.011	−0.010	−0.021	−0.001	−0.000
1.6	−0.050	−0.008	−0.007	−0.015	−0.001	−0.000
1.8	−0.034	−0.006	−0.005	−0.011	−0.000	−0.000

$\gamma = {}^5/_3$, $\varepsilon = 1$, $\omega = 0.1$, $J = 0.5$, $M = 5$.

TABLE 4

Imag (c). Odd eigenvalues. Hartmann flow

ε	0.1			10		
R	10^3	10^5	10^7	10^3	10^5	10^7
K						
0.2	−0.611	−0.030	−0.017	−69.5	−3.04	−1.69
0.4	−0.115	−0.006	−0.003	−12.4	−0.570	−0.319
0.6	−0.039	−0.002	−0.001	−3.88	−0.191	−0.107
0.8	−0.018	−0.001	−0.001	−1.75	−0.087	−0.049
1.0	−0.010	−0.001	−0.000	−0.946	−0.047	−0.026
1.2	−0.006	−0.000	−0.000	−0.577	−0.028	−0.016
1.4	−0.004	−0.000	−0.000	−0.382	−0.018	−0.011
1.6	−0.003	−0.000	−0.000	−0.268	−0.013	−0.007
1.8	−0.002	−0.000	−0.000	−0.196	−0.009	−0.005

$\gamma = {}^5/_3$, $\beta = 1000$, $\omega = 0.1$, $J = 0.5$, $M = 5$.

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